

ON THE YAMABE CONSTANTS OF $S^2 \times \mathbb{R}^3$ AND $S^3 \times \mathbb{R}^2$

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ABSTRACT. We compare the isoperimetric profiles of $S^2 \times \mathbb{R}^3$ and of $S^3 \times \mathbb{R}^2$ with that of a round 5-sphere (of appropriate radius). Then we use this comparison to obtain lower bounds for the Yamabe constants of $S^2 \times \mathbb{R}^3$ and $S^3 \times \mathbb{R}^2$. Explicitly we show that $Y(S^3 \times \mathbb{R}^2, [g_0^3 + dx^2]) > (3/4)Y(S^5)$ and $Y(S^2 \times \mathbb{R}^3, [g_0^2 + dx^2]) > 0.63Y(S^5)$. We also obtain explicit lower bounds in higher dimensions and for products of Euclidean space with a closed manifold of positive Ricci curvature. The techniques are a more general version of those used by the same authors in [15] and the results are a complement to the work developed by B. Ammann, M. Dahl and E. Humbert to obtain explicit gap theorems for the Yamabe invariants in low dimensions.

1. INTRODUCTION

Given a conformal class $[g]$ of Riemannian metrics on a closed manifold M^n the *Yamabe constant* of $[g]$, $Y(M, [g])$, is defined as

$$Y(M, [g]) = \inf_{h \in [g]} \frac{\int_M s_h \, d\text{vol}(h)}{\text{Vol}(M, h)^{\frac{n-2}{n}}},$$

where s_h and $d\text{vol}(h)$ denote the scalar curvature and volume element of h respectively. If we denote by $p = p_n = 2n/(n-2)$ and let $h = f^{p-2}g$ we can rewrite the previous expression as

$$Y(M, [g]) = \inf_{f \in C^\infty(M)} \frac{\int_M a_n |\nabla f|^2 d\text{vol}(g) + \int_M s_g f^2 d\text{vol}(g)}{(\int_M f^p d\text{vol}(g))^{2/p}},$$

where $a_n = 4(n-1)/(n-2)$.

Then one defines the Yamabe invariant of M , $Y(M)$, as the supremum of the Yamabe constants over the family of all conformal classes of metrics on M .

By a local argument T. Aubin showed in [7] that the Yamabe constant of any conformal class of metrics on any n -dimensional manifold is bounded above by $Y(S^n, [g_0^n])$, where by g_0^n we will denote from now on the round metric of sectional curvature one on S^n . It follows that $Y(S^n) = Y(S^n, [g_0^n])$ and for any n -dimensional manifold M , $Y(M) \leq Y(S^n)$. A closed manifold M has positive Yamabe invariant if and only if it admits a metric of positive scalar curvature. In this case $Y(M) \in (0, Y(S^n)]$.

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Computing the invariant when $0 < Y(M) < Y(S^n)$ is particularly difficult and interesting. There are very few cases when this has been accomplished [2, 9, 10, 11] and only recently there has been some more general results obtaining estimates in this situation.

In this article we will first concentrate in obtaining lower bounds for the Yamabe constants of $S^2 \times \mathbb{R}^3$ and $S^3 \times \mathbb{R}^2$. We point out that for a non-compact manifold (W^n, g) of positive scalar curvature we define its Yamabe constant by

$$Y(W, g) = \inf_{f \in L_1^2(W)} \frac{a_n \int_W |\nabla f|^2 d\text{vol}(g) + \int_W s_g f^2 d\text{vol}(g)}{(\int_W f^p d\text{vol}(g))^{2/p}} = \inf_{f \in L_1^2(W)} Y_g(f).$$

We will call Y_g the Yamabe functional of (W, g) .

Computing or estimating the Yamabe constants of the Riemannian products of spheres and Euclidean spaces is very important in the study of the Yamabe invariant. One main reason for this is that they play a fundamental role in understanding the behavior of the invariant under surgery. For instance they appear explicitly in the surgery formula in [3]. To obtain our lower bounds we will use the techniques we developed in [15]. The principal motivation to consider the particular cases of $S^2 \times \mathbb{R}^3$ and $S^3 \times \mathbb{R}^2$ is the recent work by B. Ammann, M. Dahl and E. Humbert [4, 5, 6] where the authors obtain an explicit gap theorem: using the estimates in this paper they show in [6] (among other things) that for any simply connected closed 5-manifold M^5 , $Y(M^5) \in (45.1, Y(S^5)]$ (note that $Y(S^5) = 78.997\dots$).

Our estimates will be obtained using appropriate lower bounds on isoperimetric profiles. Let us recall that for a Riemannian manifold (M, g) of volume V the isoperimetric function (or isoperimetric profile) of (M, g) is the function $I_{(M, g)} : (0, V) \rightarrow (0, \infty)$ given by

$$I_{(M, g)}(t) = \inf\{Vol(\partial U) : Vol(U) = t\}.$$

The principal tool to obtain our lower bounds is the following theorem (a special case of which was used in our previous article [15]):

Theorem 1.1. *Let (M^k, g) be a closed Riemannian manifold with scalar curvature $s_g \geq k(k-1)$. If $I_{(M^k \times \mathbb{R}^n, g+dx^2)}$ is a non-decreasing function and $I_{(M^k \times \mathbb{R}^n, g+dx^2)} \geq \lambda I_{(S^{n+k}, \mu g_0^{n+k})}$ then $Y(M^k \times \mathbb{R}^n, [g+dx^2]) \geq \min\{\frac{\mu k(k-1)}{(k+n)(k+n-1)}, \lambda^2\} Y(S^{n+k})$.*

It is not necessary that $I_{(M^k \times \mathbb{R}^n, g+dx^2)}$ is non-decreasing. One only needs a reasonable lower bound for the isoperimetric function on large values of the volume (after $I_{(S^{n+k}, \mu g_0^{n+k})}$ attains its maximum). For instance one could ask that $I_{(M^k \times \mathbb{R}^n, g+dx^2)}(t)$ is bounded below by the maximum of $\lambda I_{(S^{n+k}, \mu g_0^{n+k})}$ for $t \geq (1/2)Vol(S^{n+k}, \mu g_0^{n+k})$. But we are going to apply the theorem to non-compact manifolds of non-negative Ricci curvature (for which the isoperimetric profile is non-decreasing by [8, Page 52]) and this seems a more natural condition.

To apply the previous result we obtain the following estimates for the isoperimetric profiles of $(S^2 \times \mathbb{R}^3, g_0^2 + dx^2)$ and $(S^3 \times \mathbb{R}^2, g_0^3 + dx^2)$.

Theorem 1.2. $I_{(S^2 \times \mathbb{R}^3, g_0^2 + dx^2)} \geq \frac{3\sqrt{7}}{10} I_{(S^5, (63/10)g_0^5)}.$

Theorem 1.3. $I_{(S^3 \times \mathbb{R}^2, g_0^3 + dx^2)} \geq \frac{\sqrt{3}}{2} I_{(S^5, (5/2)g_0^5)}.$

Then we obtain as a corollary that:

Theorem 1.4. $Y(S^2 \times \mathbb{R}^3, [g_0^2 + dx^2]) \geq 0.63 Y(S^5)$ and $Y(S^3 \times \mathbb{R}^2, [g_0^3 + dx^2]) \geq 0.75 Y(S^5).$

The previous theorems also give lower bounds for the Yamabe invariants of certain products of manifolds. For any Riemannian manifold (M^k, g) and any n -dimensional closed manifold of positive scalar curvature (N^n, h) it is proven in [1, Theorem 1.1] that

$$\lim_{r \rightarrow \infty} Y(N^n \times M^k, [h + rg]) = Y(N^n \times \mathbb{R}^k, [h + dx^2]).$$

Therefore we also obtain as a corollary that

Theorem 1.5. *If M is a closed 3-dimensional manifold then $Y(S^2 \times M) \geq 0.63 Y(S^5)$ and if S is any closed 2-manifold then $Y(S^3 \times S) \geq 0.75 Y(S^5).$*

In Section 5 we will also find explicit lower bounds for $Y(S^7 \times \mathbb{R}^2, g_0^7 + dx^2)$ and $Y(S^8 \times \mathbb{R}^2, g_0^8 + dx^2)$. These are needed to obtain the explicit lower bounds for the Yamabe constants of compact spin manifolds in dimensions 9 and 10 in [6, Corollary 5.4]. In this case we will simplify a little the calculations, at the expense of not getting the best possible lower bounds. We do so in order to avoid an excessive number of calculations. We obtain:

Theorem 1.6. $Y(S^7 \times \mathbb{R}^2, [g_0^7 + dx^2]) \geq 0.747 Y(S^9)$ and $Y(S^8 \times \mathbb{R}^2, [g_0^8 + dx^2]) \geq 0.626 Y(S^{10}).$

One could use the previous estimates to obtain results in more general situations. For instance for a Riemannian manifold (M^k, g) of positive Ricci curvature the Levy-Gromov isoperimetric inequality compares the isoperimetric profile of (M, g) with that of the round k -sphere: if $\text{Ricci}(g) \geq (k-1)g$ and $V = \text{Vol}(M, g)$ then $I_{(M, g)}(t) \geq (V/V_k)I_{(S^k, g_0^k)}((V_k/V)t)$, where V_k is the volume of the round k -sphere.

Then applying the Ros product Theorem (see [16, Theorem 22] or [12, Section 3]) we have (using the same simple arguments we will use in Corollary 3.2 in this article) that

$$\begin{aligned} I_{(M \times \mathbb{R}^n, g + dx^2)}(t) &\geq (V/V_k)I_{(S^k \times \mathbb{R}^n, g_0^k + dx^2)}((V_k/V)t). \\ \text{If } I_{(S^k \times \mathbb{R}^n, g_0^k + dx^2)} &\geq \lambda I_{(S^{k+n}, \mu g_0^{k+n})} \text{ then we have} \\ I_{(M \times \mathbb{R}^n, g + dx^2)}(t) &\geq (V/V_k)\lambda I_{(S^{k+n}, \mu g_0^{k+n})}((V_k/V)t) \\ &= (V/V_k)\lambda(V/V_k)^{(1-(k+n))/(k+n)} I_{(S^{k+n}, \mu(V/V_k)^{2/(n+k)} g_0^{k+n})}(t) \\ &= \lambda(V/V_k)^{1/(k+n)} I_{(S^{k+n}, \mu(V/V_k)^{2/(n+k)} g_0^{k+n})}(t) \end{aligned}$$

We deduce from Theorem 1.1 that:

Theorem 1.7. *Let (M^k, g) be a closed Riemannian manifold with Ricci curvature $\text{Ricci}(g) \geq (k-1)g$ and volume V . Assume that $I_{(S^k \times \mathbb{R}^n, g_0^k + dx^2)} \geq \lambda I_{(S^{k+n}, \mu g_0^{k+n})}$. Then $Y(M^k \times \mathbb{R}^n, [g + dx^2]) \geq \min\left\{\frac{\mu(V/V_k)^{2/(k+n)}k(k-1)}{(k+n)(k+n-1)}, (\lambda(V/V_k)^{1/(k+n)})^2\right\} Y(S^{n+k})$.*

Example: Consider (\mathbf{HP}^2, g) where g is the usual Einstein metric normalized to have scalar curvature 56. Then its volume is (see the computations in [6, Appendix C]) $V = V_8 \times (2^8/7^3) \approx V_8 \times 0.746$. We will prove in Section 5 (Corollary 5.2) that $I_{(S^8 \times \mathbb{R}^2, g_0^8 + dx^2)} \geq 0.92 \times 0.86 I_{(S^{10}, (2^2/8)(2^2/9)(g_0^{10}))} = 0.7912 I_{(S^{10}, (1.387)(g_0^{10}))}$.

Then the previous theorem says that

$$Y(\mathbf{HP}^2 \times \mathbb{R}^2, [g + dx^2]) \geq (2^8/7^3)^{1/5} \min\left\{\frac{1.387 \times 56}{90}, 0.7912^2\right\} Y(S^{10}) > 0.59Y(S^{10}).$$

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2. THE ISOPERIMETRIC PROFILE OF CYLINDERS

The isoperimetric profile of the cylinders $(S^n \times \mathbb{R}, g_0^n + dx^2)$, $n \geq 2$, are known. They have been studied by R. Pedroza in [14]. Pedroza shows that isoperimetric regions are either a cylindrical section or congruent to a ball type region and gives explicit formulae for the volumes and areas of the (ball type) isoperimetric regions and their boundaries. The ball type regions Ω_h^n are balls whose boundary is a smooth sphere of constant mean curvature h . The sections of Ω_h^n , namely $\Omega_h^n \cap (S^n \times \{a\})$, are geodesic balls in S^n centered at some fixed point. If we let $\eta \in (0, \pi)$ be the maximum of the radius of those balls then $h = h_{n-1}(\eta) = \frac{(\text{Sin}(\eta))^{n-1}}{\int_0^\eta (\text{Sin}(s))^{n-1} ds}$. These ball type regions are the isoperimetric regions for small values of the volume. The formulas for the volumes of Ω_h and its boundary obtained by Pedroza are

$$(1) \quad A(\eta) = \text{Vol}(\partial\Omega_h^n) = 2V_{n-1} \int_0^\eta \frac{(\text{Sin}(y))^{n-1}}{\sqrt{1 - u_{n-1}(\eta, y)^2}} dy,$$

$$(2) \quad V(\eta) = \text{Vol}(\Omega_h^n) = 2V_{n-1} \int_0^\eta \frac{\int_0^y (\text{Sin}(s))^{n-1} ds}{\sqrt{1 - u_{n-1}(\eta, y)^2}} u_{n-1}(\eta, y) dy,$$

where

$$u_{n-1}(\eta, y) = \frac{(\text{Sin}(\eta))^{n-1} / \int_0^\eta (\text{Sin}(s))^{n-1} ds}{(\text{Sin}(y))^{n-1} / \int_0^y (\text{Sin}(s))^{n-1} ds}.$$

3. ESTIMATING THE ISOPERIMETRIC PROFILE OF $S^3 \times \mathbb{R}^2$

In this section we will prove Theorem 1.3. We will first deal with small values of the volume. Note that for any (closed or homogeneous) Riemannian n-manifold (M^n, g) one has

$$\lim_{v \rightarrow 0} \frac{I_{(M,g)}(v)}{v^{\frac{n-1}{n}}} = \gamma_n,$$

where γ_n is the classical n-dimensional isoperimetric constant:

$$\gamma_n = \frac{\text{Vol}(S^{n-1}, g_0^{n-1})}{\text{Vol}(B^n(0, 1), dx^2)^{\frac{n-1}{n}}}.$$

In particular $\gamma_4 = 2^{7/4}\sqrt{\pi}$ and $\gamma_5 = (8\pi^2/3)^{1/5}5^{4/5}$.

Lemma 3.1. $I_{(S^3 \times \mathbb{R}, g_0^3 + dx^2)} \geq 0.99 I_{(S^4, 2^{2/3}g_0^4)}.$

Proof. We first check the inequality for $v \leq 0.03$. Using formulas (1) and (2), direct computation shows that $\frac{I_{(S^3 \times \mathbb{R}, g_0^3 + dx^2)}(0.03)}{(0.03)^{3/4}} \approx 5.904 > 5.902 \approx (0.99)\gamma_4 = 0.99 \lim_{v \rightarrow 0} \frac{I_{(S^4, 2^{2/3}g_0^4)}(v)}{v^{3/4}}.$

On the other hand, we know by a theorem of V. Bayle [8, page 52] that both $\frac{I_{(S^4, 2^{2/3}g_0^4)}(v)}{v^{3/4}}$ and $\frac{I_{(S^3 \times \mathbb{R}, g_0^3 + dx^2)}(v)}{v^{3/4}}$ are decreasing (since both $(S^4, 2^{2/3}g_0^4)$ and $(S^3 \times \mathbb{R}, g_0^3 + dx^2)$ have non-negative Ricci curvature). Then it follows that for $0 \leq v \leq 0.03$

$$I_{(S^3 \times \mathbb{R}, g_0^3 + dx^2)}(v) \geq \frac{I_{(S^3 \times \mathbb{R}, g_0^3 + dx^2)}(0.03)}{(0.03)^{3/4}} v^{3/4} > (0.99)\gamma_4 v^{3/4} \geq (0.99)I_{(S^4, 2^{2/3}g_0^4)}(v).$$

The inequality for $v \geq 0.03$, can be verified using standard numerical computations, based on formulas (1) and (2). We provide the graphics (fig. 1). Note that for $v \geq v_0 \approx 20.8576$ a cylindrical section $S^3 \times [a, b]$ of volume v is isoperimetric in $(S^3 \times \mathbb{R}, g_0^3 + dx^2)$ and its boundary has volume $4\pi^2 > 0.99 4\pi^2$ which is the maximum of $0.99 I_{(S^4, 2^{2/3}g_0^4)}$. So one only needs to check the inequality for $v \leq v_0$. □

Corollary 3.2. $I_{(S^3 \times \mathbb{R}^2, g_0^3 + dx^2)} \geq 0.99 I_{(S^4 \times \mathbb{R}, 2^{2/3}g_0^4 + dx^2)} = 0.99 I_{(S^4 \times \mathbb{R}, 2^{2/3}(g_0^4 + dx^2))}.$

Proof. Ros product Theorem (see [16, Theorem 22] or [12, Section 3] says that if one has a *model* measure space (as the Euclidean spaces or the spheres of any radius) (M_0, μ_0) and any other measure spaces (M_1, μ_1) , (M_2, μ_2) such that $I_2 \geq I_0$ then $I_{\mu_1 \otimes \mu_2} \geq I_{\mu_1 \otimes \mu_0}$. If (M_0, μ_0) is a model measure with isoperimetric profile I_0 then λI_0 is also the isoperimetric profile of a model measure (obtained by changing the distance on M_0) for any positive λ . The corollary then clearly follows from Ros product Theorem and the previous lemma. □

In the next section we will use the following

Corollary 3.3. $I_{(S^3 \times \mathbb{R}^2, 2(g_0^3 + dx^2))} \geq 0.99 I_{(S^4 \times \mathbb{R}, 2^{5/3}(g_0^4 + dx^2))}.$

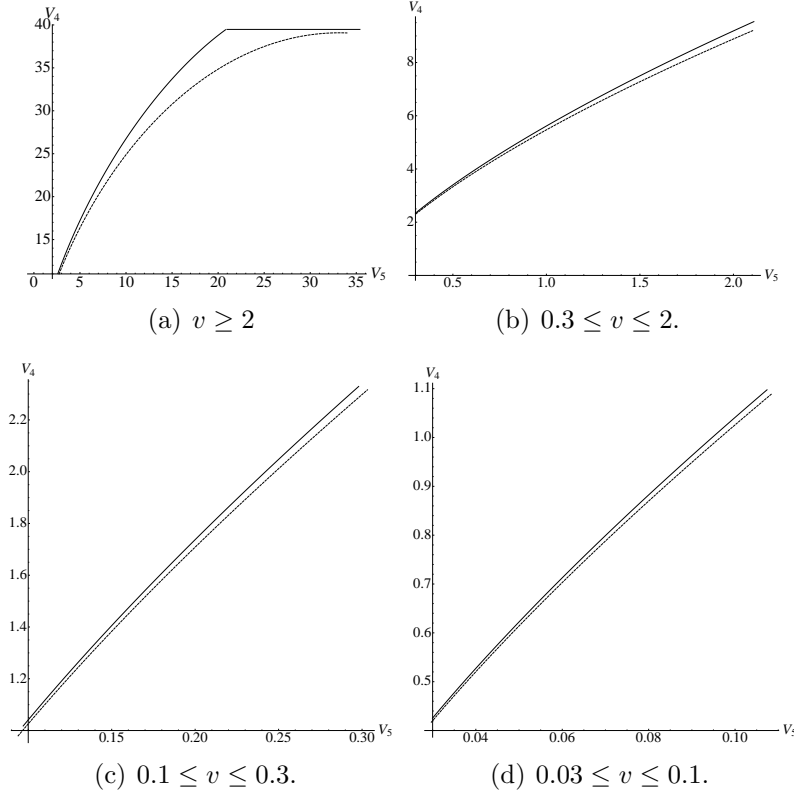


FIGURE 1. $I_{(S^3 \times \mathbb{R}, g_0^3 + dt^2)}(v) \geq I_{(S^4, 2^{2/3}g_0^4)}(v)$, for $v \geq 0.03$.

Lemma 3.4. For $v \leq 80$, $I_{(S^4 \times \mathbb{R}, 2^{2/3}g_0^4 + dt^2)}(v) \geq \sqrt{\frac{3}{4}}(0.99)^{-1} I_{(S^5, (5/2)g_0^5)}(v)$.

Proof. We begin by proving the inequality for $v \leq 4$. By direct computation, using formulas (1) and (2), we get $\frac{I_{(S^4 \times \mathbb{R}, 2^{2/3}g_0^4 + dt^2)}(4)}{(4)^{4/5}} \approx 6.2585 > 6.0971 \approx \frac{\sqrt{3}}{2}(0.99)^{-1}\gamma_5 = \frac{\sqrt{3}}{2}(0.99)^{-1} \lim_{v \rightarrow 0} \frac{I_{(S^5, \frac{5}{2}g_0^5)}(v)}{v^{4/5}}$.

By the result of Bayle mentioned above [8, page 52], the functions $\frac{I_{(S^5, \frac{5}{2}g_0^5)}(v)}{v^{4/5}}$ and $\frac{I_{(S^4 \times \mathbb{R}, 2^{2/3}(g_0^4 + dt^2))}(v)}{v^{4/5}}$ are decreasing. Hence

$$\begin{aligned} I_{(S^4 \times \mathbb{R}, 2^{2/3}(g_0^4 + dt^2))}(v) &\geq \frac{I_{(S^4 \times \mathbb{R}, 2^{2/3}(g_0^4 + dt^2))}(4)}{(4)^{4/5}} v^{4/5} > \frac{\sqrt{3}}{2}(0.99)^{-1}\gamma_5 v^{4/5} \\ &\geq \frac{\sqrt{3}}{2}(0.99)^{-1} I_{(S^5, \frac{5}{2}g_0^5)}(v), \end{aligned}$$

for $0 \leq v \leq 4$.

We now check the inequality for $4 \leq v \leq 80$, using standard numerical computations, based on formulas (1) and (2). We provide the graphics (fig. 2).

□

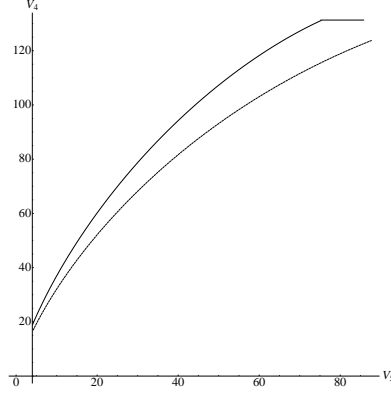


FIGURE 2. $I_{(S^4 \times \mathbb{R}, 2^{2/3}g_0^4 + dt^2)}(v) \geq \frac{\sqrt{3}}{2}(0.99)^{-1}I_{(S^5, \frac{5}{2}g_0^5)}(v)$, for $4 \leq v \leq 80$.

Lemma 3.5. For $v \geq 16$, $I_{(S^3 \times \mathbb{R}^2, g_0^3 + dx^2)}(v) \geq \frac{(2\pi)^{3/2}}{\sqrt{2}}\sqrt{v}$.

Proof. Let f_1 and f_2 be the isoperimetric profiles for (S^3, g_0^3) and (\mathbb{R}^2, dx^2) respectively. Isoperimetric regions in (S^3, g_0^3) are geodesic balls and then $f_1(v_1(t)) = 4\pi \sin^2(t)$, where $v_1(t) = 2\pi(t - \cos(t)\sin(t))$ ($t \in [0, \pi]$ and hence $v_1 \in [0, 2\pi^2]$). Isoperimetric regions in (\mathbb{R}^2, dx^2) are also geodesic balls, and so we have $f_2(t) = 2\sqrt{\pi}\sqrt{t}$.

Now consider the isoperimetric function for product regions in $(S^3 \times \mathbb{R}^2, g_0^3 + dx^2)$; $I_P(v) = \inf\{f_1(v_1)v_2 + f_2(v_2)v_1 : v_1v_2 = v\}$, which can be rewritten as

$$I_P(v) = \inf \left(\frac{2\sin^2(t)v}{t - \cos(t)\sin(t)} + 2\sqrt{\pi}\sqrt{v}\sqrt{2\pi(t - \cos(t)\sin(t))} : t \in (0, \pi) \right).$$

By a result of F. Morgan [13, Theorem 2.1] we have that $I_{(S^3 \times \mathbb{R}^2, g_3 + dx^2)}(v) \geq \frac{I_P(v)}{\sqrt{2}}$. Hence, verifying that $I_P(v) \geq (2\pi)^{3/2}\sqrt{v}$, for $v \geq 16$, will yield the Lemma. For that purpose, consider

$$F_v(t) = 2\sqrt{v} \left(\frac{\sin^2(t)\sqrt{v}}{t - \cos(t)\sin(t)} + \pi\sqrt{2(t - \cos(t)\sin(t))} \right),$$

and let $v \geq 16$. Then

$$F_v(t) \geq 2\sqrt{v} \left(\frac{4\sin^2(t)}{(t - \cos(t)\sin(t))} + \pi\sqrt{2(t - \cos(t)\sin(t))} \right).$$

But it is easy to check that $\frac{4\sin^2(t)}{(t - \cos(t)\sin(t))} + \pi\sqrt{2(t - \cos(t)\sin(t))} \geq \pi^{3/2}\sqrt{2}$, for $t \in (0, \pi)$ (the minimum is achieved at π). Then $I_P(v) \geq (2\pi)^{3/2}\sqrt{v}$, and the lemma follows. \square

Lemma 3.6. $I_{(S^3 \times \mathbb{R}^2, g_0^3 + dx^2)}(v) \geq \sqrt{\frac{3}{4}}I_{(S^5, \frac{5}{2}g_0^5)}(v)$, for $v \geq 80$.

Proof. Using again the theorem of Bayle [8, page 52], we know that $I_{(S^3 \times \mathbb{R}^2, g_0^3 + dx^2)}$ is concave. Of course, this implies that any line connecting two values of known lower bounds for $I_{(S^3 \times \mathbb{R}^2, g_0^3 + dx^2)}$ is also a lower bound for the isoperimetric function. In particular, the line $l(v) = 131.312 + 0.280204(v - 75.517)$, which joins the point $(75.517, 131.312)$ in the graphic of $0.99 I_{(S^4 \times \mathbb{R}, 2^{2/3}g_0^4 + dt^2)}(v)$ and the point $(450, 630\pi^{3/2}\sqrt{2})$ in the graphic of $\frac{(2\pi)^{3/2}}{\sqrt{2}}\sqrt{v}$, is a lower bound for $I_{(S^3 \times \mathbb{R}^2, (g_0^3 + dx^2))}$ (fig. 3). Finally, standard numerical computations show that this line is also an upper bound for $\sqrt{\frac{3}{4}}I_{(S^5, \frac{5}{2}g_0^5)}$, for $v \geq 80$ (fig. 3), and hence $I_{(S^3 \times \mathbb{R}^2, (g_3 + dx^2))} \geq \sqrt{\frac{3}{4}}I_{(S^5, \frac{5}{2}g_0^5)}(v)$, for $v \geq 80$. □

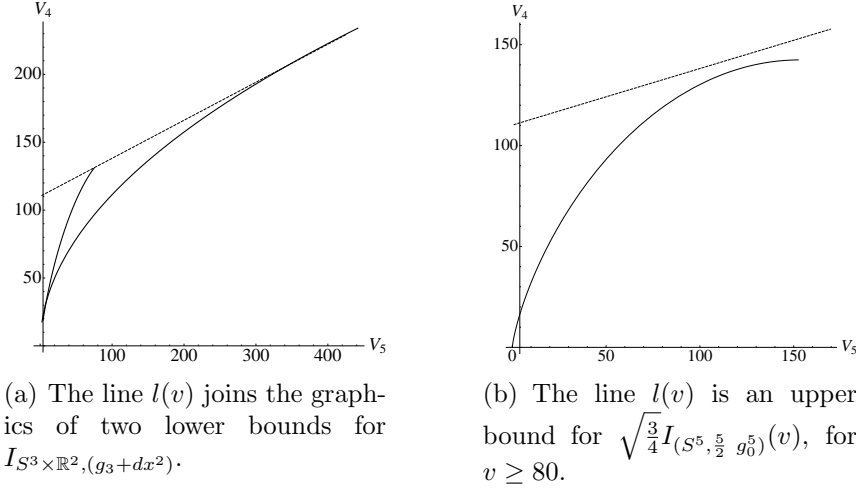


FIGURE 3. $I_{(S^3 \times \mathbb{R}^2, (g_3 + dx^2))} \geq \sqrt{\frac{3}{4}}I_{(S^5, \frac{5}{2}g_0^5)}(v)$, for $v \geq 80$.

Corollary 3.2, Corollary 3.4 and Lemma 3.6 complete the proof of Theorem 1.3.

4. ESTIMATING THE ISOPERIMETRIC PROFILE OF $S^2 \times \mathbb{R}^3$

In this section we will prove Theorem 1.2. The isoperimetric function of (S^5, g_0^5) is given by $I_{(S^5, g_0^5)}(2\pi^2((1/3)\cos^3(r) - \cos(r) + (2/3))) = (8/3)\pi^2 \sin^4(r)$. And so $\frac{3\sqrt{7}}{10}I_{(S^5, 6.3g_0^5)}((6.3)^{5/2}(2\pi^2((1/3)\cos^3(r) - \cos(r) + (2/3)))) = \frac{3\sqrt{7}}{10}(6.3)^2(8/3)\pi^2 \sin^4(r)$. The first observation is that the maximum of $\frac{3\sqrt{7}}{10}I_{(S^5, (63/10)g_0^5)}$ is $\frac{3\sqrt{7}}{5}(63/10)^2 \text{Vol}(S^4) = \frac{3\sqrt{7}}{10}(63/10)^2(8/3)\pi^2 \approx 829.12$ and is achieved at $v = (1/2)(63/10)^{5/2} \text{Vol}(S^5) = (1/2)(63/10)^{5/2}\pi^3 \approx 1544.44$. After this value of v the function $\frac{3\sqrt{7}}{10}I_{(S^5, (63/10)g_0^5)}$ is decreasing while $I_{(S^2 \times \mathbb{R}^3, g_0^2 + dx^2)}$ is always non-decreasing. It follows that to prove Theorem 1.2 we only need to consider the case $v \leq 1544.44$.

Lemma 4.1. $I_{(S^2 \times \mathbb{R}^3, g_0^2 + dx^2)} \geq 0.99 I_{(S^4 \times \mathbb{R}, 2^{5/3}(g_0^4 + dx^2))}$.

Proof. We know from [15], section 2.1, that $I_{(S^2 \times \mathbb{R}, g_0^2 + dx^2)} \geq I_{(S^3, 2g_0^3)}$. This implies using Ros product theorem [16, 12] that $I_{(S^2 \times \mathbb{R}^2, g_0^2 + dx^2)} \geq I_{(S^3 \times \mathbb{R}, 2g_0^3 + dx^2)} = I_{(S^3 \times \mathbb{R}, 2(g_0^3 + dx^2))}$. Then by using again the Ros product theorem one gets

$$I_{(S^2 \times \mathbb{R}^3, g_0^2 + dx^2)} \geq I_{(S^3 \times \mathbb{R}^2, 2(g_0^3 + dx^2))}.$$

But by Corollary 2.4 $I_{(S^3 \times \mathbb{R}^2, 2(g_0^3 + dx^2))} \geq 0.99 I_{(S^4 \times \mathbb{R}, 2^{5/3}(g_0^4 + dx^2))}$, and the lemma follows. \square

We now prove the following.

Lemma 4.2. $I_{(S^4 \times \mathbb{R}, 2^{5/3}(g_0^4 + dx^2))}(v) \geq \frac{3\sqrt{7}}{9.9} I_{(S^5, (63/10)g_0^5)}(v)$, for $v \leq 427$. And so Theorem 1.2 is true for $v \leq 427$.

Proof. We begin by proving the inequality for $v \leq 100$. Direct computation using (1) and (2) shows that $\frac{I_{(S^4 \times \mathbb{R}, 2^{5/3}(g_0^4 + dt^2))}(100)}{100^{4/5}} \approx 5.6106 > 5.5881 \approx \frac{3\sqrt{7}}{9.9} \gamma_5 = \lim_{v \rightarrow 0} \frac{3\sqrt{7}}{9.9} \frac{I_{(S^5, \frac{63}{10}g_0^5)}(v)}{v^{4/5}}$. Since $(S^5, \frac{63}{10}g_0^5)$ and $(S^4 \times \mathbb{R}, 2^{5/3}(g_0^4 + dt^2))$ have non-negative Ricci curvature it follows from [8] that both $\frac{I_{(S^5, \frac{32}{5}g_0^5)}(v)}{v^{4/5}}$ and $\frac{I_{(S^4 \times \mathbb{R}, 2^{5/3}(g_0^4 + dt^2))}(v)}{v^{4/5}}$ are decreasing. Therefore

$$I_{(S^4 \times \mathbb{R}, 2^{5/3}(g_0^4 + dt^2))}(v) \geq \frac{I_{(S^4 \times \mathbb{R}, 2^{5/3}(g_0^4 + dt^2))}(100)}{(100)^{4/5}} v^{4/5} > \frac{3\sqrt{7}}{9.9} \gamma_5 v^{4/5} \geq \frac{3\sqrt{7}}{9.9} \gamma_5 I_{(S^5, \frac{63}{10}g_0^5)}(v),$$

for $0 \leq v \leq 100$.

Next, we check the inequality for $100 \leq v \leq 427$, using standard numerical computations, based on formulas (1) and (2). We provide the graphics (fig. 4).

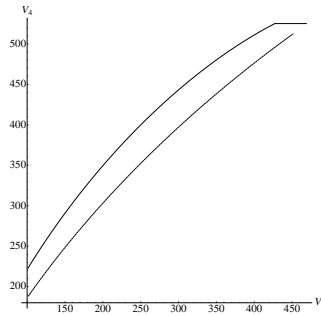


FIGURE 4. $I_{(S^4 \times \mathbb{R}, 2^{5/3}(g_0^4 + dt^2))}(v) \geq \frac{4}{5} I_{(S^5, \frac{32}{5}g_0^5)}(v)$, for $100 \leq v \leq 427$.

\square

Lemma 4.3. For $v \geq 27$, $I_{(S^2 \times \mathbb{R}^3, g_0^2 + dx^2)}(v) \geq 2^{5/6} (3\pi)^{2/3} v^{2/3}$.

Proof. Let h_1 and h_2 be the isoperimetric profiles for (S^2, g_0^2) and (\mathbb{R}^3, dx^2) respectively. Isoperimetric regions in (S^2, g_0^2) are geodesic balls and then $h_1(v_1(t)) = 2\pi \sin(t)$, where $v_1(t) = 2\pi(1 - \cos(t))$, ($t \in [0, \pi]$ and hence $v_1 \in [0, 4\pi]$). Similarly

$h_2(t) = 6^{2/3}\pi^{1/3}t^{2/3}$. Now consider the isoperimetric function for product regions in $(S^2 \times \mathbb{R}^3, g_0^2 + dx^2)$, $I_P(v) = \inf\{h_1(v_1)v_2 + h_2(v_2)v_1 : v_1v_2 = v\}$, which can be rewritten as

$$I_P(v) = \inf \left(\frac{v \sin(t)}{1 - \cos(t)} + 2(3\pi)^{2/3} \left(\frac{v}{1 - \cos(t)} \right)^{2/3} (1 - \cos(t)) : t \in (0, \pi) \right).$$

It follows from [13, Theorem 2.1] that $I_{(S^2 \times \mathbb{R}^3, g_0^2 + dx^2)}(v) \geq \frac{I_P(v)}{\sqrt{2}}$, since both I_{S^2} and $I_{\mathbb{R}^3}$ are concave. Hence, it remains to show that $I_P(v) \geq 2^{4/3}(3\pi)^{2/3}v^{2/3}$, for $v \geq 27$, to prove the lemma. For that purpose, consider

$$F_v(t) = v^{2/3} \left(\frac{v^{1/3} \sin(t)}{1 - \cos(t)} + 2(3\pi)^{2/3}(1 - \cos(t))^{1/3} \right),$$

and let $v \geq 27$. Then

$$F_v(t) \geq v^{2/3} \left(\frac{3 \sin(t)}{1 - \cos(t)} + 2(3\pi)^{2/3}(1 - \cos(t))^{1/3} \right)$$

But, as it is easy to check,

$$\frac{3 \sin(t)}{1 - \cos(t)} + 2(3\pi)^{2/3}(1 - \cos(t))^{1/3} \geq 2(2^{1/3})(3\pi)^{2/3},$$

for $t \in [0, \pi]$ (the minimum of the expresion on the left is achieved precisely at π). Hence $I_P(v) \geq 2(2^{1/3})(3\pi)^{2/3}v^{2/3}$, and the lemma follows. \square

Lemma 4.4. *Theorem 1.2 is true for $v \geq 427$.*

Proof. Since $I_{(S^2 \times \mathbb{R}^3, g_0^2 + dx^2)}$ is concave any line connecting two values of known lower bounds for $I_{(S^2 \times \mathbb{R}^3, g_0^2 + dx^2)}$ is also a lower bound for the function (between the two points). In particular, the line

$$f(v) = 525.45 + \frac{(2^{5/6}(4500\pi)^{2/3} - 525.245)(v - 427.18)}{1073},$$

which joins the point $(427.18, 525.245)$ (in the graphic of $0.99I_{(S^4 \times \mathbb{R}, 2^{5/3}(g_0^4 + dx^2))}$) and $(1500, 2^{5/6}(4500\pi)^{2/3})$ (which belongs to the graphic of $2^{5/6}(3\pi)^{2/3}v^{2/3}$), is a lower bound of $I_{(S^2 \times \mathbb{R}^3, (g_0^2 + dx^2))}$ for $v \in [427, 1500]$. Finally, standard numerical computations show that this line is also an upper bound for $\frac{3\sqrt{7}}{10}I_{(S^5, (63/10)g_0^5)}$ in the same interval (fig. 5). And this implies in particular that for $v \geq 1500$ $I_{(S^2 \times \mathbb{R}^3, g_0^2 + dx^2)}(v)$ is greater than the maximum of $\frac{3\sqrt{7}}{10}I_{(S^5, (63/10)g_0^5)}$, proving the lemma. \square

Theorem 1.2 follows from Lemma 4.2 and Lemma 4.4.

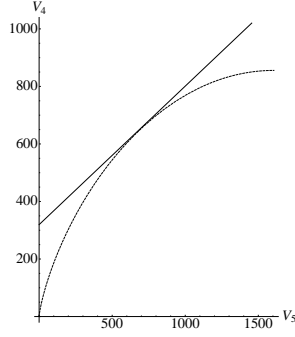


FIGURE 5. The line $f(v)$ is an upper bound for $\frac{3\sqrt{7}}{10}I_{(S^5, \frac{63}{10}g_0^5)}(v)$, for $v \geq 400$.

5. ESTIMATING THE ISOPERIMETRIC PROFILES OF $S^7 \times \mathbb{R}^2$ AND $S^8 \times \mathbb{R}^2$

We first note as in section 3 that for any (closed or homogeneous) Riemannian n -manifold (M^n, g) one has

$$\lim_{v \rightarrow 0} \frac{I_{(M,g)}(v)}{v^{\frac{n-1}{n}}} = \gamma_n,$$

where γ_n is the classical n -dimensional isoperimetric constant:

$$\gamma_n = \frac{\text{Vol}(S^{n-1}, g_0^{n-1})}{\text{Vol}(B^n(0, 1), dx^2)^{\frac{n-1}{n}}}.$$

In this section we will need the values

$$\gamma_8 = (8^7/3)^{1/8} \sqrt{\pi} \approx 9.5310,$$

$$\gamma_9 = (32\pi^4 9^8/105)^{1/9} \approx 10.2762$$

$$\text{and } \gamma_{10} = (10^9/12)^{1/10} \sqrt{\pi} \approx 10.9814.$$

Lemma 5.1. $I_{(S^7 \times \mathbb{R}, g_0^7 + dx^2)} \geq 0.94 I_{(S^8, 2^{2/7} g_0^8)}$, $I_{(S^8 \times \mathbb{R}, g_0^8 + dx^2)} \geq 0.92 I_{(S^9, 2^{1/4} g_0^9)}$ and $I_{(S^9 \times \mathbb{R}, g_0^9 + dx^2)} \geq 0.86 I_{(S^{10}, 2^{2/9} g_0^{10})}$.

Proof. We first use formulas (1) and (2), and direct computation, to find some $\alpha_n > 0$ (for $n = 7, 8, 9$) such that $\frac{I_{(S^n \times \mathbb{R}, g_0^n + dx^2)}(\alpha_n)}{(\alpha_n)^{n/(n+1)}} > (\beta_n) \gamma_{n+1} = (\beta_n) \lim_{v \rightarrow 0} \frac{I_{(S^{n+1}, 2^{2/n} g_0^{n+1})}(v)}{v^{n/(n+1)}}$ (where $\beta_7 = 0.94$, $\beta_8 = 0.92$ and $\beta_9 = 0.86$). The values of these α_n are included in the following table.

n	α_n	$\frac{I_{(S^n \times \mathbb{R}, g_0^n + dx^2)}(\alpha_n)}{(\alpha_n)^{n/(n+1)}}$	$\beta_n \gamma_{n+1}$	β_n
7	0.0052	9.04	8.96	0.94
8	0.0068	9.51	9.45	0.92
9	0.0018	9.49	9.44	0.86

Next, we use these values of α_n to prove the inequalities of the lemma for small values of v : we know by a theorem of V. Bayle [8, page 52] that both $\frac{I_{(S^{n+1}, 2^{2/n} g_0^{n+1})}(v)}{v^{n/(n+1)}}$ and

$\frac{I_{(S^n \times \mathbb{R}, g_0^n + dx)}(v)}{v^{n/n+1}}$ are decreasing (since both $(S^{n+1}, 2^{2/n} g_0^{n+1})$ and $(S^n \times \mathbb{R}, g_0^n + dx^2)$ have non-negative Ricci curvature). Then it follows that for $0 \leq v \leq \alpha_n$,

$$I_{(S^n \times \mathbb{R}, g_0^n + dx^2)}(v) \geq \frac{I_{(S^n \times \mathbb{R}, g_0^n + dx^2)}(\alpha_n)}{(\alpha_n)^{n/n+1}} v^{n/n+1} \\ > (\beta_n) \gamma_{n+1} v^{n/n+1} \geq \beta_n I_{(S^{n+1}, 2^{2/n} g_0^{n+1})}(v).$$

The inequality for $v \geq \alpha_n$, can be verified using standard numerical computations, based on formulas (1) and (2). However, since $I_{(S^n \times \mathbb{R}, g_0^n + dx^2)}$ is concave (this follows also from [8, page 52], as $(S^n \times \mathbb{R}, g_0^n + dx^2)$ has non-negative Ricci curvature) then it suffices to show that $\beta_n I_{(S^{n+1}, 2^{2/n} g_0^{n+1})}$ is bounded from above by the straight lines joining together points of $I_{(S^n \times \mathbb{R}, g_0^n + dx^2)}$. We provide the graphics for each case (figures 6, 7 and 8). Note also that for each n , there is some $v_{0,n}$, such that for $v \geq v_{0,n}$ a cylindrical section $S^n \times [a_n, b_n]$ of volume v is isoperimetric in $(S^n \times \mathbb{R}, g_0^n + dx^2)$ and its boundary has volume $2w_n > \beta_n 2w_n$ which is the maximum of $\beta_n I_{(S^{n+1}, 2^{2/n} g_0^{n+1})}$. So one only needs to check the inequality for $v \leq v_{0,n}$.

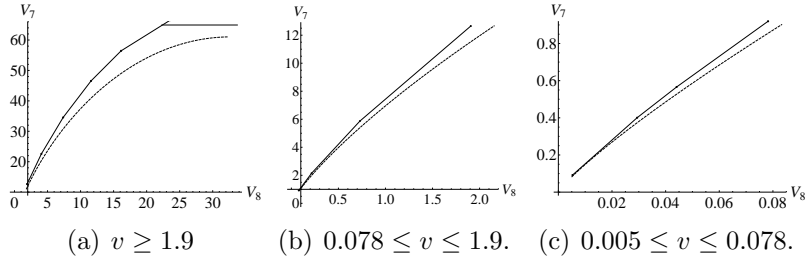


FIGURE 6. $I_{(S^7 \times \mathbb{R}, g_0^7 + dt^2)}(v) \geq 0.94 I_{(S^8, 2^{2/7} g_0^8)}(v)$, for $v \geq 0.005$.

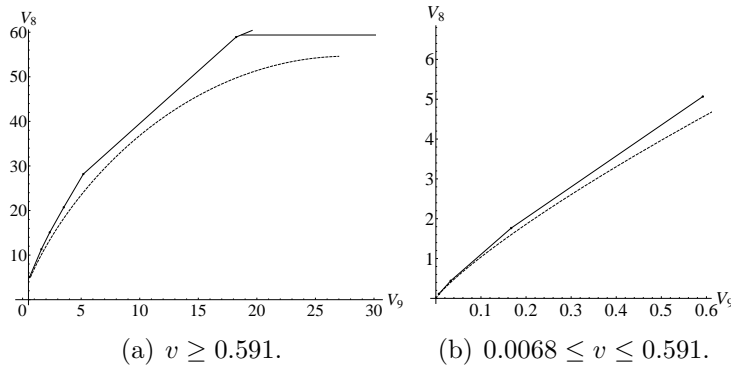


FIGURE 7. $I_{(S^8 \times \mathbb{R}, g_0^8 + dt^2)}(v) \geq 0.92 I_{(S^9, 2^{1/4} g_0^9)}(v)$, for $v \geq 0.0068$.

□

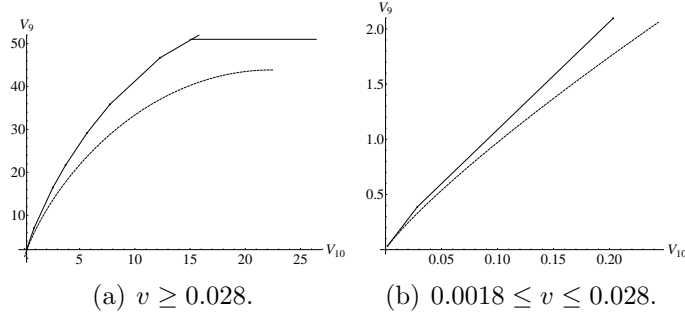


FIGURE 8. $I_{(S^9 \times \mathbb{R}, g_0^9 + dx^2)}(v) \geq 0.86 I_{(S^{10}, 2^{2/9} g_0^{10})}(v)$, for $v \geq 0.0018$.

Corollary 5.2. For $n = 7$ and $n = 8$, $I_{(S^n \times \mathbb{R}^2, g_0^n + dx^2)} \geq \beta_n \beta_{n+1} I_{(S^{n+2}, (2^{2/n})^{(2^{2/(n+1)})}(g_0^{n+2}))}$.

Proof. The previous lemma tells us that for $n = 7$ and $n = 8$ $I_{(S^n \times \mathbb{R}, g_0^n + dx^2)} \geq \beta_n I_{(S^{n+1}, 2^{2/n} g_0^{n+1})}$. Then the same argument as in the proof of Corollary 3.2 implies that $I_{(S^n \times \mathbb{R}^2, g_0^n + dx^2)} \geq \beta_n I_{(S^{n+1} \times \mathbb{R}, 2^{2/n} g_0^{n+1} + dx^2)} = \beta_n I_{(S^{n+1} \times \mathbb{R}, 2^{2/n} (g_0^{n+1} + dx^2))}$. From the previous lemma it follows that $I_{(S^{n+1} \times \mathbb{R}, 2^{2/n} (g_0^{n+1} + dx^2))} \geq \beta_{n+1} I_{(S^{n+2}, (2^{2/n})^{(2^{2/(n+1)})}(g_0^{n+2}))}$ and the corollary follows. \square

Using the previous corollary and Theorem 1.1 we have

$$Y(S^7 \times \mathbb{R}^2, g_0^7 + dx^2) \geq \min \left\{ \frac{42 \times 2^{2/7+1/4}}{72}, (\beta_7 \beta_8)^2 \right\} Y(S^9) = \min\{0.845, 0.747\} Y(S^9).$$

And

$$Y(S^8 \times \mathbb{R}^2, g_0^8 + dx^2) \geq \min \left\{ \frac{56 \times 2^{2/9+1/4}}{90}, (\beta_8 \beta_9)^2 \right\} Y(S^{10}) = \min\{0.863, 0.626\} Y(S^{10}).$$

6. PROOF OF THEOREM 1.1

Proof. This is a general version of what appears in [15, Theorem 1.2]. The proof is essentially the same, we give the details for completeness.

Let $f : M^k \times \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ be any smooth compactly supported function.

First assume that $\text{Vol}(\{f > 0\}) \leq \text{Vol}(S^{n+k}, \mu g_0^{n+k})$. Let $f_* : (S^{n+k}, \mu g_0^{n+k}) \rightarrow \mathbb{R}_{\geq 0}$ be the spherical symmetrization of f : f_* is a radial (it depends only on the distance to some fixed point in S^{n+k}), non-increasing function on the sphere such that for any $t > 0$, $\text{Vol}(\{f > t\}) = \text{Vol}(\{f_* > t\})$ (here the volume is measured with respect to the volume element of μg_0^{n+k}). We want to compare the values of the (corresponding) Yamabe functional in f and f_* . It is immediate that for any $q > 0$, $\|f\|_q = \|f_*\|_q$ and we need to compare the L^2 -norm of the gradients.

By the coarea formula

$$\int \|\nabla f\|^2 d\text{vol}(g + dx^2) = \int_0^\infty \left(\int_{f^{-1}(t)} \|\nabla f\| d\sigma_t \right) dt,$$

where $d\sigma_t$ denotes the volume element of the induced metric on $f^{-1}(t)$. And by Hölder's inequality

$$\int_0^\infty \left(\int_{f^{-1}(t)} \|\nabla f\| d\sigma_t \right) dt \geq \int_0^\infty (\text{Vol}(f^{-1}(t)))^2 \left(\int_{f^{-1}(t)} \|\nabla f\|^{-1} d\sigma_t \right)^{-1} dt.$$

But, applying the coarea formula again,

$$\int_{f^{-1}(t)} \|\nabla f\|^{-1} d\sigma_t = -\frac{d}{dt}(\{f > t\}) = -\frac{d}{dt}(\text{Vol}(\{f_* > t\})) = \int_{f_*^{-1}(t)} \|\nabla f_*\|^{-1} d\sigma_t.$$

Since $f^{-1}(t)$ contains the boundary of $\{f > t\}$ and $\text{Vol}(\{f > t\}) = \text{Vol}(\{f_* > t\})$ (which is an isoperimetric region in the sphere), it follows that $\text{Vol}(f^{-1}(t)) \geq \text{Vol}(\partial(\{f > t\})) \geq \lambda \text{Vol}(f_*^{-1}(t))$. Then using that $\|\nabla f_*\|$ is constant along level surfaces of f_* and the coarea formula

$$\begin{aligned} \int \|\nabla f\|^2 d\text{vol}(g + dx^2) &\geq \lambda^2 \int_0^\infty (\text{Vol}(f_*^{-1}(t)))^2 \left(\int_{f_*^{-1}(t)} \|\nabla f_*\|^{-1} d\sigma_t \right)^{-1} dt \\ &= \lambda^2 \int_0^\infty \text{Vol}(f_*^{-1}(t)) \|\nabla f_*\| dt = \lambda^2 \int_0^\infty \left(\int_{f_*^{-1}(t)} \|\nabla f_*\| d\sigma_t \right) dt \\ &= \lambda^2 \int \|\nabla f_*\|^2 d\text{vol}(\mu g_0^{n+k}). \end{aligned}$$

Finally we have

$$\begin{aligned} Y_{g+dx^2}(f) &= \frac{a_{k+n} \int_{M \times \mathbb{R}^n} \|\nabla f\|^2 d\text{vol}(g + dx^2) + \int_{M \times \mathbb{R}^n} s_g f^2 d\text{vol}(g + dx^2)}{(\int_{M \times \mathbb{R}^n} f^{p_{k+n}} d\text{vol}(g + dx^2))^{2/p_{k+n}}} \\ &\geq \frac{a_{k+n} \lambda^2 \int_{S^{k+n}} \|\nabla f_*\|^2 d\text{vol}(\mu g_0^{k+n}) + \int_{S^{k+n}} k(k-1) f_*^2 d\text{vol}(\mu g_0^{k+n})}{(\int_{S^{k+n}} f_*^{p_{k+n}} d\text{vol}(\mu g_0^{k+n}))^{2/p_{k+n}}} \\ &\geq \min \left(\lambda^2, \frac{\mu k(k-1)}{(k+n)(k+n-1)} \right) \times \\ &\quad \frac{a_{k+n} \int_{S^{k+n}} \|\nabla f_*\|^2 d\text{vol}(\mu g_0^{k+n}) + \int_{S^{k+n}} (k+n)(k+n-1)(1/\mu) f_*^2 d\text{vol}(\mu g_0^{k+n})}{(\int_{S^{k+n}} f_*^{p_{k+n}} d\text{vol}(\mu g_0^{k+n}))^{2/p_{k+n}}} \\ &= \min \left(\lambda^2, \frac{\mu k(k-1)}{(k+n)(k+n-1)} \right) Y_{\mu g_0^{k+n}}(f_*). \end{aligned}$$

Now assume that $Vol(\{f > 0\}) > Vol(S^{n+k}, \mu g_0^{n+k})$. Then let $t_0 = \max(f)$ and pick $t_0 > t_1 \geq t_2 > \dots > t_N = 0$ such that for $i = 1, \dots, N-1$ we have that $Vol(f^{-1}(t_i, t_{i-1})) = Vol(S^{n+k}, \mu g_0^{n+k})$ and $Vol(f^{-1}(0, t_{N-1})) \leq Vol(S^{n+k}, \mu g_0^{n+k})$. We let f_i be the restriction of f to $f^{-1}(t_i, t_{i-1})$ and $f_{i*} : (S^{n+k}, \mu g_0^{n+k}) \rightarrow [t_i, t_{i-1}]$ be its radial symmetrization (as above). Since $I_{(M^k \times \mathbb{R}^n, g+dx^2)}$ is non-decreasing we can use essentially the same argument as before to obtain

$$\begin{aligned} \int_{f^{-1}(t_i, t_{i-1})} \|\nabla f\|^2 dvol(g+dx^2) &= \int_{f^{-1}(t_i, t_{i-1})} \|\nabla f_i\|^2 dvol(g+dx^2) \\ &\geq \lambda^2 \int_{f^{-1}(t_i, t_{i-1})} \|\nabla f_{i*}\|^2 dvol(\mu g_0^{n+k}) \end{aligned}$$

Finally,

$$\begin{aligned} Y_{g+dx^2}(f) &= \frac{a_{k+n} \int_{M \times \mathbb{R}^n} \|\nabla f\|^2 dvol(g+dx^2) + \int_{M \times \mathbb{R}^n} s_g f^2 dvol(g+dx^2)}{(\int_{M \times \mathbb{R}^n} f^{p_{k+n}} dvol(g+dx^2))^{2/p_{k+n}}} \\ &\geq \frac{\sum_{i=1}^N (a_{k+n} \lambda^2 \int_{S^{k+n}} \|\nabla f_{i*}\|^2 dvol(\mu g_0^{k+n}) + \int_{S^{k+n}} k(k-1) f_{i*}^2 dvol(\mu g_0^{k+n}))}{(\sum_{i=1}^N \int_{S^{k+n}} f_{i*}^{p_{k+n}} dvol(\mu g_0^{k+n}))^{2/p_{k+n}}} \\ &\geq \min \left(\lambda^2, \frac{\mu k(k-1)}{(k+n)(k+n-1)} \right) \times \\ &\quad \frac{\sum_{i=1}^N (a_{k+n} \int_{S^{k+n}} \|\nabla f_{i*}\|^2 dvol(\mu g_0^{k+n}) + \int_{S^{k+n}} (k+n)(k+n-1)(1/\mu) f_{i*}^2 dvol(\mu g_0^{k+n}))}{(\sum_{i=1}^N \int_{S^{k+n}} f_{i*}^{p_{k+n}} dvol(\mu g_0^{k+n}))^{2/p_{k+n}}} \\ &\geq \min \left(\lambda^2, \frac{\mu k(k-1)}{(k+n)(k+n-1)} \right) \frac{\sum_{i=1}^N Y(S^{k+n}) (\int_{S^{k+n}} f_{i*}^{p_{k+n}} dvol(\mu g_0^{k+n}))^{2/p_{k+n}}}{(\sum_{i=1}^N \int_{S^{k+n}} f_{i*}^{p_{k+n}} dvol(\mu g_0^{k+n}))^{2/p_{k+n}}} \end{aligned}$$

(since $Y(S^{k+n})$ is the Yamabe constant of $(S^{k+n}, \mu g_0^{k+n})$)

$$\geq \min \left(\lambda^2, \frac{\mu k(k-1)}{(k+n)(k+n-1)} \right) Y(S^{k+n})$$

(since $x^{2/p_{k+n}} + y^{2/p_{k+n}} \geq (x+y)^{2/p_{k+n}}$, $x, y \geq 0$). And this concludes the proof of the theorem.

□

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